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I.

FINITE DIFFERENCE APPROXIMATION OF THE DEFLECTION EQUATIONS

OF A CONICAL SHELL WITH CLOSED BASE

By

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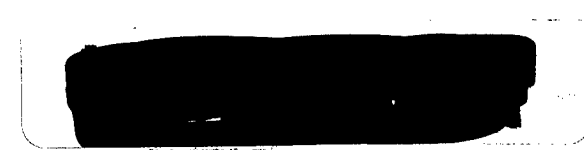


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## I. INTRODUCTION

In this report we give finite difference approximations to the differential equations for the following problems:

(1) the bending of a variable-thickness truncated right conical shell attached to an axially and sectionally symmetric base plate of variable thickness, (2) the bending of two joined variable-thickness truncated right conical shells.

In case (1) above, the terms arising from consideration of the transverse deflection of the base caused by radial loading have been included. As a consequence the finite difference equations are non-linear. For solutions of these equations we suggest an iterative scheme combined with a direct method based on the factorization theorem for square matrices. In case (2), the finite difference equations are linear and procedures for solution are straightforward.

## II. DIFFERENTIAL EQUATIONS

The differential equations for the conical shells are derived in TR 1-21 [1]\* and are repeated below for convenience. Coordinate system, geometry, and variables are shown in Fig. 1. The differential equations are given in terms of the variables  $S$  and  $\theta$  where  $S$  is related to the force per unit length  $N_\theta$  (see Fig. 2) by:

\* Numbers in square brackets refer to bibliography on Page 29.

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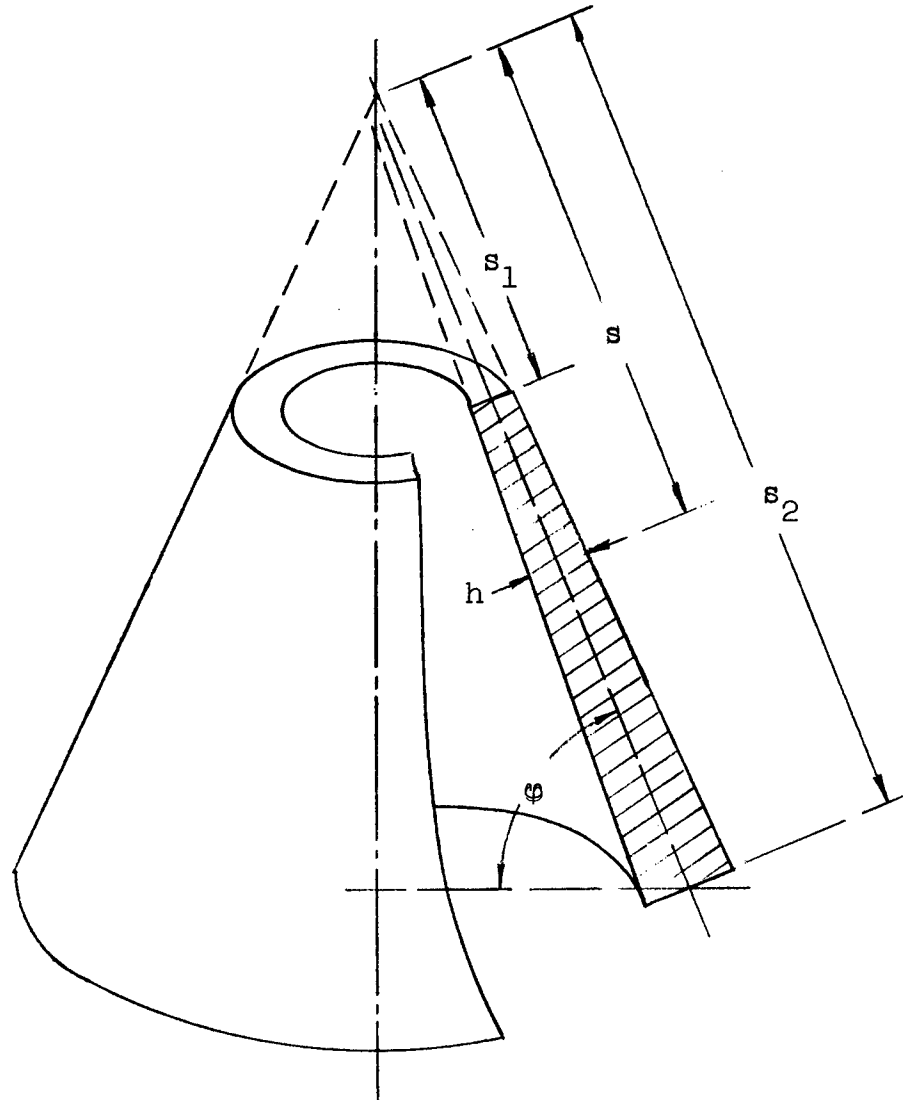


Fig. 1

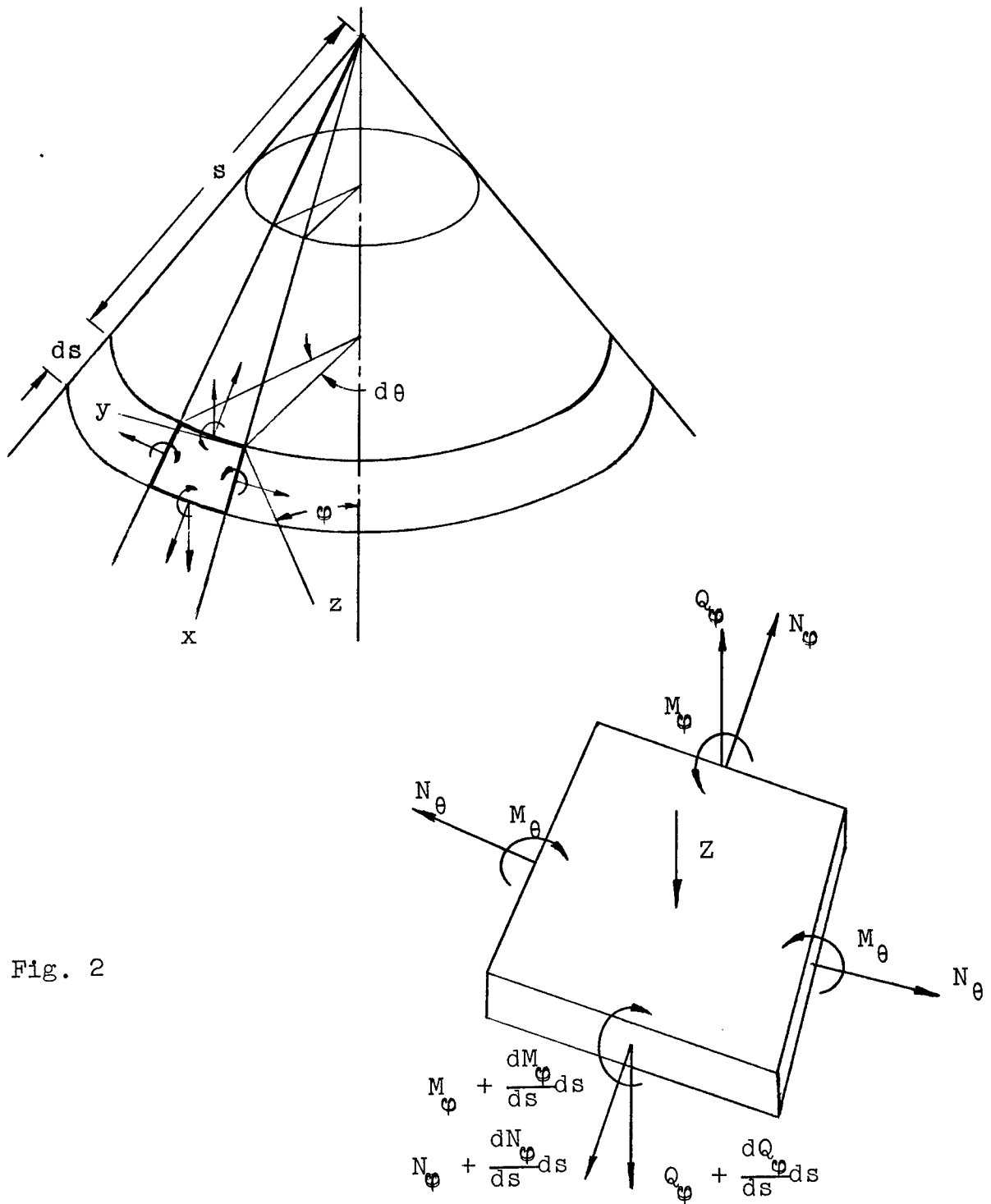


Fig. 2

$$2.1 \quad S = \frac{sN}{h^2}$$

and  $\theta$  is the slope of the median plane in the axial direction,  
i.e.:

$$2.2 \quad \theta = \frac{dw}{ds}$$

The differential equations are ([1] p. 16):

$$2.3 \quad L(\theta) + f_1\theta = \lambda_1 S + F(s)$$

$$2.4 \quad L(S) + f_2 S = \lambda_2 \theta$$

where the differential operator  $L$  is of second order:

$$2.5 \quad L(U) = h \cot\varphi \left[ sU'' + \left(1 + 3s \frac{h'}{h}\right)U' - \frac{U}{s} \right]$$

The apostrophe represents differentiation with respect to  $s$ .  
The other quantities appearing in the differential equations  
are defined as follows:

$$2.6 \quad f_1 = svh' \cot\varphi$$

where  $\nu$  is Poisson's ratio,

$$2.7 \quad f_2 = [(2 + \nu)h' + 2sh''] \cot \varphi$$

$$2.8 \quad \lambda_1 = \frac{12(1 - \nu^2)}{E}$$

where E is the modulus of elasticity,

$$2.9 \quad \lambda_2 = -E$$

and

$$2.10 \quad F(s) = \frac{12(1 - \nu^2)}{E} \frac{F(s)}{h^2 \cot \varphi}$$

where ([1] p. 4):

$$F(s) = - [sN_\varphi \sin \varphi + s Q_\varphi \cos \varphi] = \int_{s_0}^s Z \cos \varphi \, s \, ds$$

$$2.11$$

$$- [sN_\varphi \sin \varphi + sQ_\varphi \cos \varphi]_{s=s_0}$$

and Z is a distributed load normal to the middle surface of the shell and acting over the whole surface, and  $Q_\varphi$  is a shear force per unit length acting normal to the middle surface of the shell (see Fig.2)

The differential equations for the base may be written as follows [2]:

$$2.12 \quad (D\phi')' + \frac{D}{r} \phi' + \left( \frac{v}{r} D - \frac{D}{r^2} - \sigma_r(a)t \right) \phi = Q(r)$$

$$2.13 \quad r^2 \psi'' + \left( 1 - \frac{r}{t} t' \right) r \psi' + \left( \frac{vr}{t} t' - 1 \right) \psi = 0$$

where the apostrophe represents differentiation with respect to  $r$  (the radial distance from the center of the plate),  $t(r)$  is the thickness of the plate,  $D = E(t(r))^3/12(1 - \nu^2)$ ,  $v(r)$  is the transverse deflection of the plate,  $\sigma_r(r)$  is the membrane stress in the plate,  $q$  is a uniformly distributed load acting transversely upon the plate, and

$$2.14 \quad Q(r) = \frac{qr}{2}$$

$$2.15 \quad \frac{v(a)}{r} = \sigma_r(a)t(r)$$

and

$$2.16 \quad \phi = \frac{dv}{dr}$$

Coordinate system, geometry, and variables are shown in Fig. 3.

For the case of the bending of two joined cones the differential equation for each cone is given by corresponding sets of equations 2.3 and 2.4 along with definitions 2.5, 2.6, 2.7, 2.8, 2.9, 2.10 and 2.11.



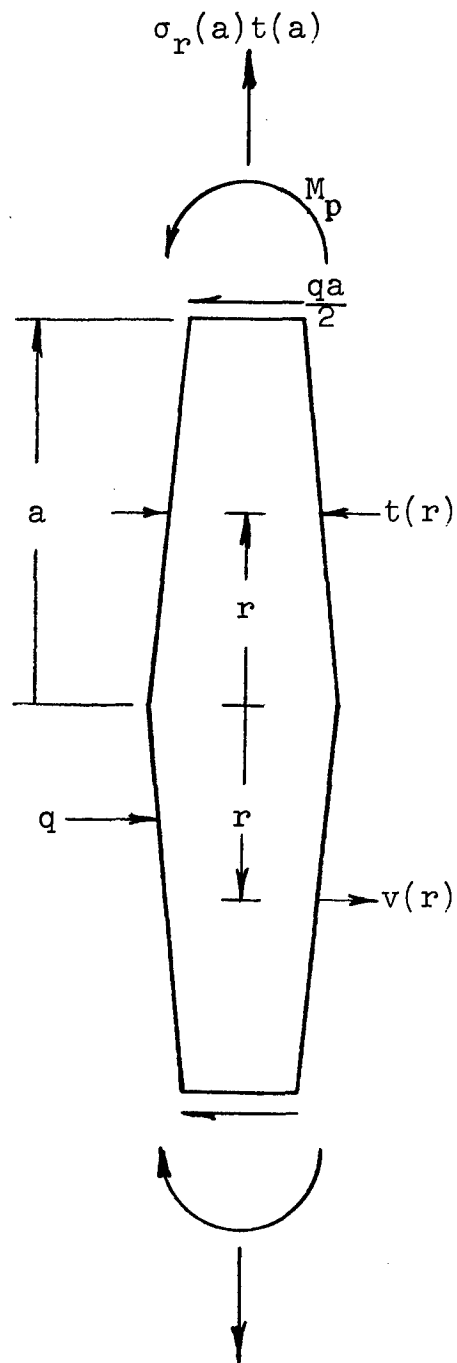


Fig. 3

DIAMETRAL SECTION OF BASE-PLATE

## III. BOUNDARY AND JUNCTION CONDITIONS

For the cone with attached base-plate the boundary conditions at the free end of the cone are

$$3.1 \quad M_{\phi} = M \quad \text{at open end of cone}$$

$$3.2 \quad \epsilon_{ym} = \epsilon$$

and at the center of the base-plate are

$$3.3 \quad \phi = 0 \quad \text{at center of base-plate}$$

$$3.4 \quad \psi = 1$$

where  $M_{\phi}$  is moment per unit length and  $\epsilon_{ym}$  is the strain of the middle surface of the cone in the  $y$  direction (circumferentially).

In terms of the variables of 2.3 and 2.4 the boundary conditions 3.1 and 3.2 may be written

$$3.5 \quad \frac{Eh^3}{12(1 - \nu^2)} \left( \theta' + \frac{\nu}{s}\theta \right) = M$$

at open end of cone

$$3.6 \quad h(2h' - \frac{\nu h}{s})S + h^2 S' = \epsilon$$

At the junction of the cone and the base plate the following conditions may be prescribed

$$\left. \begin{aligned}
 3.7 \quad \sigma_r t &= V \\
 3.8 \quad M_p &= M_n \\
 3.9 \quad -\phi &= \theta \\
 3.10 \quad \epsilon_p &= \epsilon_{ym}
 \end{aligned} \right\} \text{at the junction of the cone and base-plate}$$

where  $M_p$  and  $\epsilon_p$  are respectively the moment per unit length and the strain along the circumference of the base-plate.

In terms of the variables of 2.3, 2.4, 2.12 and 2.13 the junction conditions can be written

$$\left. \begin{aligned}
 3.11 \quad \frac{\psi}{r} &= P \tan \rho - \frac{h^2 S}{s \cos \rho} \\
 3.12 \quad \frac{Et^3}{12(1-\nu^2)} (\phi' + \frac{\nu}{r} \phi) &= \frac{Eh^3}{12(1-\nu^2)} (\theta' + \frac{\nu}{s} \theta) \\
 3.13 \quad -\phi &= \theta \\
 3.14 \quad \frac{1}{Et} (\psi' - \frac{\nu}{r} \psi) &= h(2h' - \frac{\nu h}{s}) S + h^2 S'
 \end{aligned} \right\} \text{at junction of cone and base-plate}$$

where  $P = Q = \frac{qa}{2}$

For the case of the joined cones the conditions at the free ends are

$$3.15 \quad M_{\phi 1} = M_1$$

$$3.16 \quad M_{\phi 2} = M_2$$

$$3.17 \quad \epsilon_{ym1} = \epsilon_1$$

$$3.18 \quad \epsilon_{ym2} = \epsilon_2$$

and at the joined ends are ([1], p. 20)

$$3.19 \quad \left[ (2hh' - v \frac{h^2}{s})s + h^2 s' \right]_1 = \left[ (2hh' - v \frac{h^2}{s})s + h^2 s' \right]_2$$

$$3.20 \quad \theta_1 = \theta_2$$

$$3.21 \quad \left[ \frac{Eh^3}{12(1 - \nu^2)} (\theta' + \nu \frac{\theta}{s}) \right]_1 = \left[ \frac{Eh^3}{12(1 - \nu^2)} (\theta' + \nu \frac{\theta}{s}) \right]_2$$

$$3.22 \quad (P \tan \phi - \frac{h^2 s}{s \cos \phi})_1 = (P \tan \phi - \frac{h^2 s}{s \cos \phi})_2$$

where the subscripts 1 and 2 refer to respective joined cones.

## IV. FINITE DIFFERENCE SCHEME

We will approximate the differential equations for the cone at equally spaced net points  $s_0, s_1, s_2, \dots, s_I$  where

$$s_{i+1} - s_i = \Delta s$$

We will approximate the differential equations for the base plate by equally spaced net points  $r_0, r_1, r_2, \dots, r_J$  where

$$r_{j+1} - r_j = \Delta r$$

We will append a subscript to each variable to designate its corresponding approximation at a point. For example,  $S(s_i) = S_i$ ,  $\phi(r_j) = \phi_j$ , and similarly for the remaining variables. Derivatives will be obtained by a Taylor's series expansion about surrounding net points, neglecting terms  $O((\Delta s)^2)$  and  $O((\Delta r)^2)$ .

The difference approximations for the cone equations are

$$4.1 \quad L(\theta_1) + F_{11} \theta_1 = \lambda_1 S_1 + F_1$$

$$4.2 \quad L(S_1) + F_{21} S_1 = \lambda_2 \theta_1$$

where

$$4.3 \quad F_{11} = s_1 \nu \frac{h_{i+1} - h_{i-1}}{2\Delta s} \cot \alpha$$

$$4.4 \quad F_{21} = \left[ (2 + \nu) \frac{h_{1+1} - h_{1-1}}{2\Delta s} + (2s_1) \frac{h_{1-1} - 2h_1 + h_{1+1}}{(\Delta s)^2} \right] \cot \eta$$

$$4.5 \quad F_1 = \frac{12(1 - \nu^2)}{E} \sum_{s=s_0}^{s_1} Z \cos \eta \, s \Delta s - s_0 \left[ \frac{(N_\eta)_0 \sin \eta}{h_0^2 \cot \eta} + \frac{(Q_\eta)_0 \cos \eta}{h_0^2 \cot \eta} \right]$$

and the difference approximation for the differential operator is

$$4.6 \quad L(U_1) = h_1 \cot \eta \left[ s_1 U_1'' + \left( 1 + 3s_1 \frac{h_1'}{h_1} \right) U_1' - \frac{U_1}{s_1} \right]$$

The first and second derivatives are approximated as usual by

$$4.7 \quad V_1' = \frac{V_{1+1} - V_{1-1}}{2\Delta s} + O((\Delta s)^2)$$

and

$$4.8 \quad V_1'' = \frac{V_{1-1} - 2V_1 + V_{1+1}}{(\Delta s)^2} + O((\Delta s)^2)$$

Substituting 4.7 and 4.8 in 4.6 gives

$$L(U_1) = h_1 \cot \eta \left[ \left\{ \frac{s_1}{(\Delta s)^2} - \left( 1 + \frac{3s_1(h_{1+1} - h_{1-1})}{2\Delta s} \right) \frac{1}{2\Delta s} \right\} U_{1-1} \right]$$

$$\begin{aligned}
 4.9 \quad & - \left\{ \frac{2s_1}{(\Delta s)^2} + \frac{1}{s_1} \right\} U_1 \\
 & + \left\{ \frac{s_1}{(\Delta s)^2} + \left( 1 + \frac{3s_1(h_{1+1} - h_{1-1})}{2\Delta s} \right) \frac{1}{2\Delta s} \right\} U_{1+1} ]
 \end{aligned}$$

The difference approximations for the base-plate equations are

$$4.10 \quad (D_j \phi_j')' + \frac{D_j}{r_j} \phi_j' + \left( \frac{v}{r_j} D_j' - \frac{D_j}{r_j^2} - (\sigma_r)_J t_j \right) \phi_j = Q_j$$

$$4.11 \quad r_j^2 \phi_j'' + (1 - \frac{r_j}{t_j} t_j') r_j \phi_j' + \left( \frac{v r_j}{t_j} t_j' - 1 \right) \phi_j = 0$$

where

$$4.12 \quad D_j = \frac{E t_j^3}{12(1 - \nu^2)}$$

Substituting 4.7 and 4.8 in 4.10 and 4.11 gives

$$\begin{aligned}
 & \left\{ \frac{D_{j+1} - D_{j-1}}{(2\Delta r)^2} + \frac{D_j}{(\Delta r)^2} + \frac{D_j}{2r_j \Delta r} \right\} \phi_{j+1} \\
 4.13 \quad & + \left\{ \frac{v}{r_j} \frac{D_{j+1} - D_{j-1}}{2\Delta r} - \frac{D_j}{r_j^2} - \sigma_r(a) t_j - \frac{2D_j}{(\Delta r)^2} \right\} \phi_j
 \end{aligned}$$

$$+ \left\{ \frac{D_{j-1} - D_{j+1}}{(2\Delta r)^2} + \frac{D_j}{(\Delta r)^2} - \frac{D_j}{2r_j \Delta r} \right\} \phi_{j-1} = Q_j$$

and

$$\left\{ \frac{r_j^2}{(\Delta r)^2} + \left( 1 - \frac{r_j}{t_j} \left[ \frac{t_{j+1} - t_{j-1}}{2\Delta r} \right] \right) \frac{r_j}{2\Delta r} \right\} \psi_{j+1}$$

$$4.14 \quad + \left\{ \frac{r_j}{t_j} \left[ \frac{t_{j+1} - t_{j-1}}{2\Delta r} \right] - 1 - \frac{2r_j^2}{(\Delta r)^2} \right\} \psi_j$$

$$+ \left\{ \frac{r_j^2}{(\Delta r)^2} - \left( 1 - \frac{r_j}{t_j} \left[ \frac{t_{j+1} - t_{j-1}}{2\Delta r} \right] \right) \frac{r_j}{2\Delta r} \right\} \psi_{j-1} = 0$$

For the case of the cone with attached base-plate we first investigate the difference approximations of the differential equations at their boundaries and junction, i.e., where  $i = 0$  and  $I$ , and where  $j = 0$  and  $J$ .

For the cone when  $i = 0$  the variables  $\theta_{-1}$ ,  $\theta_0$ ,  $\theta_1$ , and  $S_0$  occur in equation 4.1 and the variables  $S_{-1}$ ,  $S_0$ ,  $S_1$ , and  $\theta_0$  occur in equation 4.2. These two relations, i.e., 4.1 and 4.2 when  $i = 0$ , will be denoted

$$4.15 \quad H_1(\theta_{-1}, \theta_0, \theta_1, S_0) = k_1$$

and

$$4.16 \quad H_2(S_{-1}, S_0, S_1, \theta_0) = k_2$$

For the base-plate when  $j = J$  the variables  $\phi_{J-1}$ ,  $\phi_J$ ,



$\phi_{J+1}$  occur in equation 4.10 and the variables  $\psi_{J-1}$ ,  $\psi_J$ ,  $\psi_{J+1}$  occur in equation 4.11. These two relations, i.e., 4.10 and 4.11 when  $j = J$ , will be denoted

$$4.17 \quad H_3(\phi_{J-1}, \phi_J, \phi_{J+1}) = k_3$$

and

$$4.18 \quad H_4(\psi_{J-1}, \psi_J, \psi_{J+1}) = k_4$$

In 4.15 and 4.16 the variables  $\theta_{-1}$  and  $S_{-1}$  are located outside the junction of the cone and base-plate; similarly in 4.17 and 4.18 the variables  $\phi_{J+1}$  and  $\psi_{J+1}$  are located outside the junction of the cone and base-plate. These four variables can be eliminated by means of the junction conditions 3.12 and 3.14 which in difference form for  $i = 0$ ,  $j = J$  are

$$4.19 \quad \frac{Et_j^3}{12(1 - \nu^2)} \left( \frac{\phi_{J+1} - \phi_{J-1}}{2\Delta r} + \frac{\nu}{a} \phi_J \right) = \frac{E h_o^3}{12(1 - \nu^2)} \left( \frac{\theta_1 - \theta_{-1}}{2\Delta s} + \frac{\nu}{s_o} \theta_o \right)$$

$$4.20 \quad \frac{1}{Et_j} \left( \frac{\psi_{J+1} - \psi_{J-1}}{2\Delta r} - \frac{\nu}{a} \psi_J \right) = h_o s_o \left( 2 \frac{h_1 - h_{-1}}{2\Delta s} - \frac{\nu h_o}{s_o} \right) + h_o^2 \frac{S_1 - S_{-1}}{2\Delta s}$$

Solving 4.15 and 4.16 for  $\theta_{-1}$  and  $S_{-1}$  and substituting in the right hand side of 4.19 and 4.20 respectively and solving 4.17 and 4.18 for  $\phi_{J+1}$  and  $\psi_{J+1}$  and substituting in the left-hand side of 4.19 and 4.20 respectively gives two equations in the

variables  $\theta_0, \theta_1, \phi_J, \phi_{J-1}$  and  $S_0, S_1, \psi_J, \psi_{J-1}$ . These equations will be denoted

$$4.21 \quad H_5(\theta_0, \theta_1, \phi_J, \phi_{J-1}) = k_5$$

$$4.22 \quad H_6(S_0, S_1, \psi_J, \psi_{J-1}) = k_6$$

We next eliminate  $\theta_0$  and  $S_0$  from 4.21 and 4.22 by means of the remaining junction conditions 3.11 and 3.13. In finite difference form for  $i = 0, j = J$  these junction conditions become

$$4.23 \quad \frac{\psi_J}{a} = P \tan \epsilon_0 - \frac{h_0^2 S_0}{s_0 \cos \epsilon_0}$$

$$4.24 \quad -\phi_J = \theta_0$$

Solving 4.23 and 4.24 for  $S_0$  and  $\theta_0$  respectively and substituting in 4.22 and 4.21 respectively gives two equations in the variables  $\theta_1, \phi_{J-1}, \phi_J$  and  $S_1, \psi_{J-1}, \psi_J$ . These equations may be denoted

$$4.25 \quad H_7(\phi_J, \phi_{J-1}, \theta_1) = k_7$$

$$4.26 \quad H_8(\psi_J, \psi_{J-1}, S_1) = k_8$$

Now we turn to the boundary conditions. Setting  $i = I$  in equations 4.1 and 4.2 gives two equations which we denote as

$$4.27 \quad H_9(\theta_{I-1}, \theta_I, \theta_{I+1}, S_I) = k_9$$

$$4.28 \quad H_{10}(S_{I-1}, S_I, S_{I+1}, \theta_I) = k_{10}$$

The outside points  $\theta_{I+1}$  and  $S_{I+1}$  can be eliminated by means of the boundary conditions 3.5 and 3.6. In difference form when  $i = I$  these conditions are

$$4.29 \quad \frac{E h_I^3}{12(1 - \nu^2)} \left( \frac{\theta_{I+1} - \theta_{I-1}}{2\Delta s} + \frac{\nu}{S_I} \theta_I \right) = M$$

$$4.30 \quad h_I \left( 2 \frac{h_{I+1} - h_{I-1}}{2\Delta s} - \frac{\nu h_I}{S_I} \right) S_I + h_I^2 \left( \frac{S_{I+1} - S_{I-1}}{2\Delta s} \right) = e$$

Solving 4.29 for  $\theta_{I+1}$  and 4.31 for  $S_{I+1}$  and substituting in 4.27 and 4.28 respectively gives two equations which we will denote as

$$4.31 \quad H_{11}(\theta_I, \theta_{I-1}, S_I) = k_{11}$$

$$4.32 \quad H_{12}(S_I, S_{I-1}, \theta_I) = k_{12}$$

Inasmuch as the conditions 3.3 and 3.4 at the center of the plate are given in terms of the functions rather than their derivatives, the corresponding finite difference equations for the base-plate differential equations are written for  $j = 1$ . The equations may be denoted

$$4.33 \quad H_{13}(\phi_0, \phi_1, \phi_2) = k_{13}$$

$$4.34 \quad H_{14}(\psi_0, \psi_1, \psi_2) = k_{14}$$

The boundary conditions 3.3 and 3.4 in difference form are

$$4.35 \quad \phi_0 = 0$$

$$4.36 \quad \psi_0 = 1$$

Substituting 4.35 and 4.36 in 4.33 and 4.34 gives two equations which we will denote as

$$4.37 \quad H_{15}(\phi_1, \phi_2) = k_{15}$$

$$4.38 \quad H_{16}(\psi_1, \psi_2) = k_{16}$$

In 4.25, 4.26, 4.31, 4.32, 4.37, and 4.38 we have four equations in the four boundary variables  $\phi_1$ ,  $\psi_1$ ,  $\theta_I$ , and  $S_I$ , and two equations in the two junction variables  $\phi_J$ ,  $\psi_J$ .

We next obtain  $2(I - 1) + 2(J - 2)$  equations at interior points by evaluating 4.1 and 4.2 at  $i = 1, 2, \dots, I-1$  and 4.10 and 4.11 at  $j = 2, 3, \dots, J-1$ .

When evaluating 4.1 and 4.2 at  $i = 1$  we bring in  $\theta_0$  and  $S_0$ . In order that all points lying on the junction satisfy the junction conditions, these terms must be eliminated through 4.24 and 4.23 as was done in the case of 4.21 and 4.22. Thus for  $i = 1$  we can denote 4.1 and 4.2 as

$$4.39 \quad H_{17}(\theta_1, \theta_2, \phi_J, S_1) = k_{17}$$

$$4.40 \quad H_{18}(S_1, S_2, \psi_J, \theta_1) = k_{18}$$

The remaining  $2(I - 2)$  equations for the cone may be denoted

$$4.41 \quad G_{1i}(\theta_{i-1}, \theta_i, \theta_{i+1}, S_i) = k_{1i} \quad i = 2, 3, \dots, I-2$$

$$4.42 \quad G_{2i}(S_{i-1}, S_i, S_{i+1}, \theta_i) = k_{2i} \quad i = 2, 3, \dots, I-2$$

At the interior points of the base plate the  $2(j - 2)$  equations 4.10 and 4.11 may be denoted

$$4.43 \quad G_{3j}(\phi_{j-1}, \phi_j, \phi_{j+1}) = k_{3j} \quad j = 2, 3, \dots, J-1$$

$$4.44 \quad G_{4j}(\psi_{j-1}, \psi_j, \psi_{j+1}) = k_{4j} \quad j = 2, 3, \dots, j-1$$

We now investigate the difference approximations for the joined cones at their boundaries and junction. At the junction, cone 1 has its  $I$ th net point and cone 2 has its  $O$ th net point. Substituting  $i = 0$  and  $i = I$  in 4.1 and 4.2 gives four relations which we will denote

$$4.45 \quad K_1(\theta_{-1}, \theta_0, \theta_1, S_0) = h_1$$

$$4.46 \quad K_2(S_{-1}, S_0, S_1, \theta_0) = h_2$$

$$4.47 \quad K_3(\theta_{I-1}, \theta_I, \theta_{I+1}, S_I) = h_3$$

$$4.48 \quad K_4(S_{I-1}, S_I, S_{I+1}, \theta_I) = h_4$$

In these four equations we have  $\theta_{-1}$ ,  $S_{-1}$ ,  $\theta_{I+1}$ , and  $S_{I+1}$  which lie outside the junction. These four variables can be eliminated by means of junction conditions 3.19 and 3.21 which in difference form are

$$4.49 \quad \left[ h_I S_I \left( 2 \frac{h_{I+1} - h_{I-1}}{2\Delta s} - \frac{v h_I}{s_I} + h_I^2 \left( \frac{S_{I+1} - S_{I-1}}{2\Delta s} \right) \right]_1 \right. \\ \left. = \left[ h_o S_o \left( 2 \frac{h_1 - h_{-1}}{2\Delta s} - \frac{v h_o}{s_o} + h_o^2 \left( \frac{S_1 - S_{-1}}{2\Delta s} \right) \right) \right]_2 \right.$$

$$4.50 \quad \left[ \frac{E h_I^3}{12(1 - v^2)} \left( \frac{\theta_{I+1} - \theta_{I-1}}{2\Delta s} + \frac{v}{s_I} \theta_I \right) \right]_1 \\ = \left[ \frac{E h_o^3}{12(1 - v^2)} \left( \frac{\theta_1 - \theta_{-1}}{2\Delta s} + \frac{v}{s_o} \theta_o \right) \right]_2$$

Solving 4.46 and 4.48 for  $S_{-1}$  and  $S_{I+1}$  respectively and substituting in 4.49 gives one equation in the variables  $S_o$ ,  $S_1$ ,  $\theta_o$ ,  $S_{I-1}$ ,  $S_I$ , and  $\theta_I$ . Solving 4.45 and 4.47 for  $\theta_{-1}$  and  $\theta_{I+1}$  respectively and substituting in 4.50 gives one equation in the variables  $\theta_o$ ,  $\theta_1$ ,  $S_o$ ,  $\theta_{I-1}$ ,  $\theta_I$ , and  $S_I$ . These

two equations may be denoted

$$4.51 \quad K_5(S_0, S_1, \theta_0, S_{I-1}, S_I, \theta_I) = h_5$$

$$4.52 \quad K_6(\theta_0, \theta_1, S_0, \theta_{I-1}, \theta_I, S_I) = h_6$$

We next replace  $\theta_0$  by  $\theta_I$  through junction condition 3.20 and we replace  $S_0$  by  $S_I$  through junction condition 3.22. Equations 4.51 and 4.52 then become

$$4.53 \quad K_7(S_1, S_{I-1}, S_I, \theta_I) = h_7$$

$$4.54 \quad K_8(\theta_1, \theta_{I-1}, \theta_I, S_I) = h_8$$

At the free end of cone 1 we may denote equations 4.1 and 4.2 in finite difference form as

$$4.55 \quad K_9(\theta_{-1}, \theta_0, \theta_1, S_0) = h_9$$

$$4.56 \quad K_{10}(S_{-1}, S_0, S_1, \theta_0) = h_{10}$$

Boundary conditions 3.15 and 3.17 in finite difference form are

$$4.57 \quad \frac{E h_0^3}{12(1 - \nu^2)} \left( \frac{\theta_1 - \theta_{-1}}{2\Delta s} + \frac{\nu}{s_0} \theta_0 \right) = M_1$$

$$4.58 \quad h_0 \left( 2 \frac{h_1 - h_{-1}}{2\Delta s} - \frac{\nu h_0}{s_0} \right) S_0 + h_0^2 \left( \frac{S_1 - S_{-1}}{2\Delta s} \right) = \epsilon_1$$

Solving 4.57 and 4.58 for  $\theta_{-1}$  and  $S_{-1}$  respectively and substituting in 4.55 and 4.56 gives 2 equations which we will denote

$$4.59 \quad K_{11}(\theta_0, \theta_1, S_0) = h_{11}$$

$$4.60 \quad K_{12}(S_0, S_1, \theta_0) = h_{12}$$

At the free end of cone 2 we have two equations (see 4.31 and 4.32) which we may denote

$$4.61 \quad K_{13}(\theta_I, \theta_{I-1}, S_I) = h_{13}$$

$$4.62 \quad K_{14}(S_I, S_{I-1}, \theta_I) = h_{14}$$

In 4.53, 4.54, 4.59, 4.60, 4.61, and 4.62 we have four equations in the four boundary variables  $\theta_0$  and  $S_0$  of cone 1 and  $\theta_I$  and  $S_I$  of cone 2 and we have two equations in the two junction variables  $\theta_I$  and  $S_I$  of cone 1. We next obtain  $2(I - 1)$  equations at the interior points of cone 1 and  $2(1 - 2)$  equations at the interior points of cone 2. When evaluating 4.1 and 4.2 at  $i = 1$  we bring in, as mentioned previously,  $\theta_0$  and  $S_0$ . In the case of cone 1, however, we already have equations in  $\theta_0$  and  $S_0$  (i.e., 4.59 and 4.60) and therefore the equations at interior points may be denoted



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$$4.63 \quad F_{1i}(\theta_{i-1}, \theta_i, \theta_{i+1}, S_i) = h_{1i} \quad i = 1, 2, \dots, I-1$$

$$4.64 \quad F_{2i}(S_{i-1}, S_i, S_{i+1}, \theta_i) = h_{2i} \quad i = 1, 2, \dots, I-1$$

In the case of cone 2 we eliminate  $\theta_0$  and  $S_0$  through junction conditions 3.20 and 3.22 to give

$$4.65 \quad F_{31}(\theta_I, \theta_1, \theta_2, S_1) = h_{31}$$

$$4.66 \quad F_{4i}(\theta_{i-1}, \theta_i, \theta_{i+1}, S_i) = h_{4i} \quad i = 2, 3, \dots, I-1$$

$$4.67 \quad F_{51}(S_I, S_1, S_2, \theta_1) = h_{51}$$

$$4.68 \quad F_{6i}(S_{i-1}, S_i, S_{i+1}, \theta_i) = h_{6i} \quad i = 2, 3, \dots, I-1$$

## V. METHOD OF SOLUTION OF THE DIFFERENCE EQUATIONS

We can see the form of the difference equations for the cone and base-plate equations if we define the following matrices

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \theta_I \end{bmatrix}$$

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \phi_J \end{bmatrix}$$

$$S = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ S_I \end{bmatrix}$$

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \psi_J \end{bmatrix}$$

The equations of the previous section can thus be written in the form

$$5.1 \quad A \begin{bmatrix} \theta \\ S \\ \Phi \\ \Psi \end{bmatrix} = R$$

The matrices A and R can be formed from equations 4.25, 4.26, 4.31, 4.32, 4.37, 4.38, 4.39, 4.40, 4.41, 4.42, 4.43, and 4.44. The matrix A is symmetric and its diagonal matrices are tri-diagonal; the disposition of its elements is shown on page 27. The equations corresponding to the elements of each row are given to the left of the matrix. An asterisk indicates non-zero constant coefficients and  $\eta$  indicates the presence of the non-linear term due to the coupling introduced by 2.15. The matrices  $A_{11}$ ,  $A_{12}$ ,  $A_{21}$ ,  $A_{22}$  are square with I rows and columns; the matrices  $A_{33}$  and  $A_{44}$  are square with J rows and columns. The remaining matrices are rectangular with I by J and J by I rows and columns.

The solution of the set of equations 5.1 could be obtained easily if it were not for the non-linearity. Hence we consider an iterative solution which assumes that  $\sigma_P(a)$  in 2.15 is known. If we let the superscript n designate the nth iteration, we proceed as follows: 1) choose the initial  $\sigma_P(a)$  as  $(\sigma_P)^0$ , 2) solve the linear system and calculate a new value for  $\sigma_P(a)$  from 2.15 using  $\psi(a)$ , 3) use the new value of  $\sigma_P(a)$  to calculate a new value of  $\psi(a)$ , 4) repeat the above process until desired convergence is obtained.

We would expect this iterative scheme to converge because  $\sigma_r(a)\hat{\phi}_j$  is relatively small. If the scheme does not converge, some linear extrapolation or interpolation between successive iterations may produce convergence. We leave this question to a future investigation.

Finally we need only give a practical method of solving the linear equations that arise in the iterative scheme. Since the diagonal matrices are tri-diagonal, we can use the method of factoring the matrix into a lower and upper matrix. This factorization requires the inversion of the  $A_{11}$ ,  $A_{22}$ ,  $A_{33}$ , and  $A_{44}$ . The resulting equations can be easily solved using a digital computer [3].

In the case of the joined cones we define the following matrices

$$\theta_1 = \begin{bmatrix} (\theta_1)_0 \\ (\theta_1)_1 \\ - \\ - \\ - \\ (\theta_1)_{I_1} \end{bmatrix} \quad S_1 = \begin{bmatrix} (s_1)_0 \\ (s_1)_1 \\ - \\ - \\ - \\ (s_1)_{I_1} \end{bmatrix}$$

$$\theta_2 = \begin{bmatrix} (\theta_2)_1 \\ (\theta_2)_2 \\ - \\ - \\ - \\ (\theta_2)_{I_2} \end{bmatrix} \quad S_2 = \begin{bmatrix} (s_2)_1 \\ (s_2)_2 \\ - \\ - \\ - \\ (s_2)_{I_2} \end{bmatrix}$$

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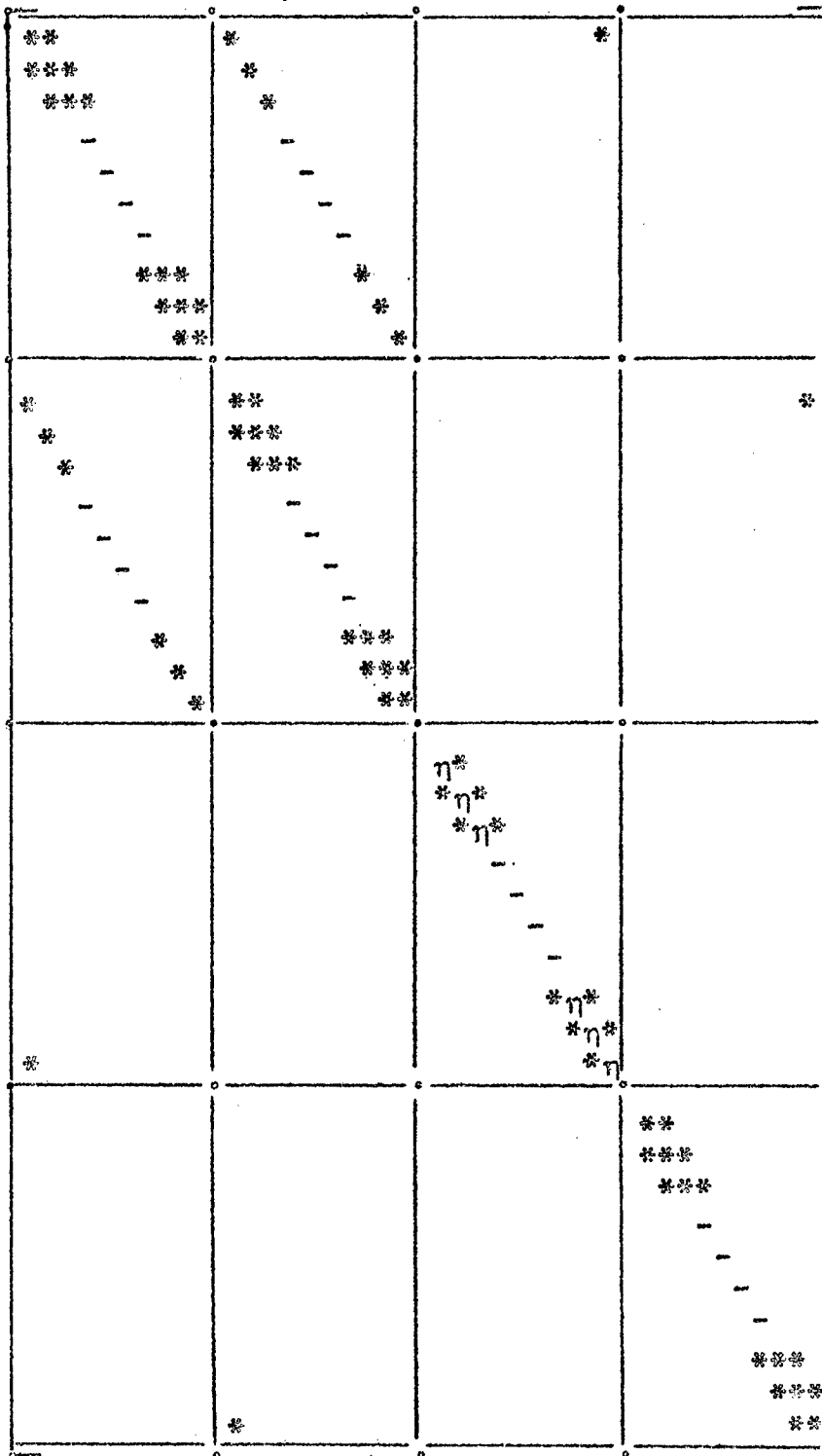
The appropriate equations can then be written

$$B \begin{bmatrix} \theta_1 \\ s_1 \\ \theta_2 \\ s_2 \end{bmatrix} = T$$

The matrices B and T can be formed from equations 4.53, 4.54, 5.49, 4.60, 4.61, 4.62, 4.63, 4.64, 4.65, 4.66, 4.67, and 4.68. The matrix B is symmetric and its diagonal matrices are tri-diagonal; the disposition of elements is shown on page 28. The equations corresponding to the elements of each row are given to the left of the matrix. An asterisk indicates non-zero constant coefficients. Since all equations are linear their solution can be carried out by a single application of the method described previously for an iterative cycle.

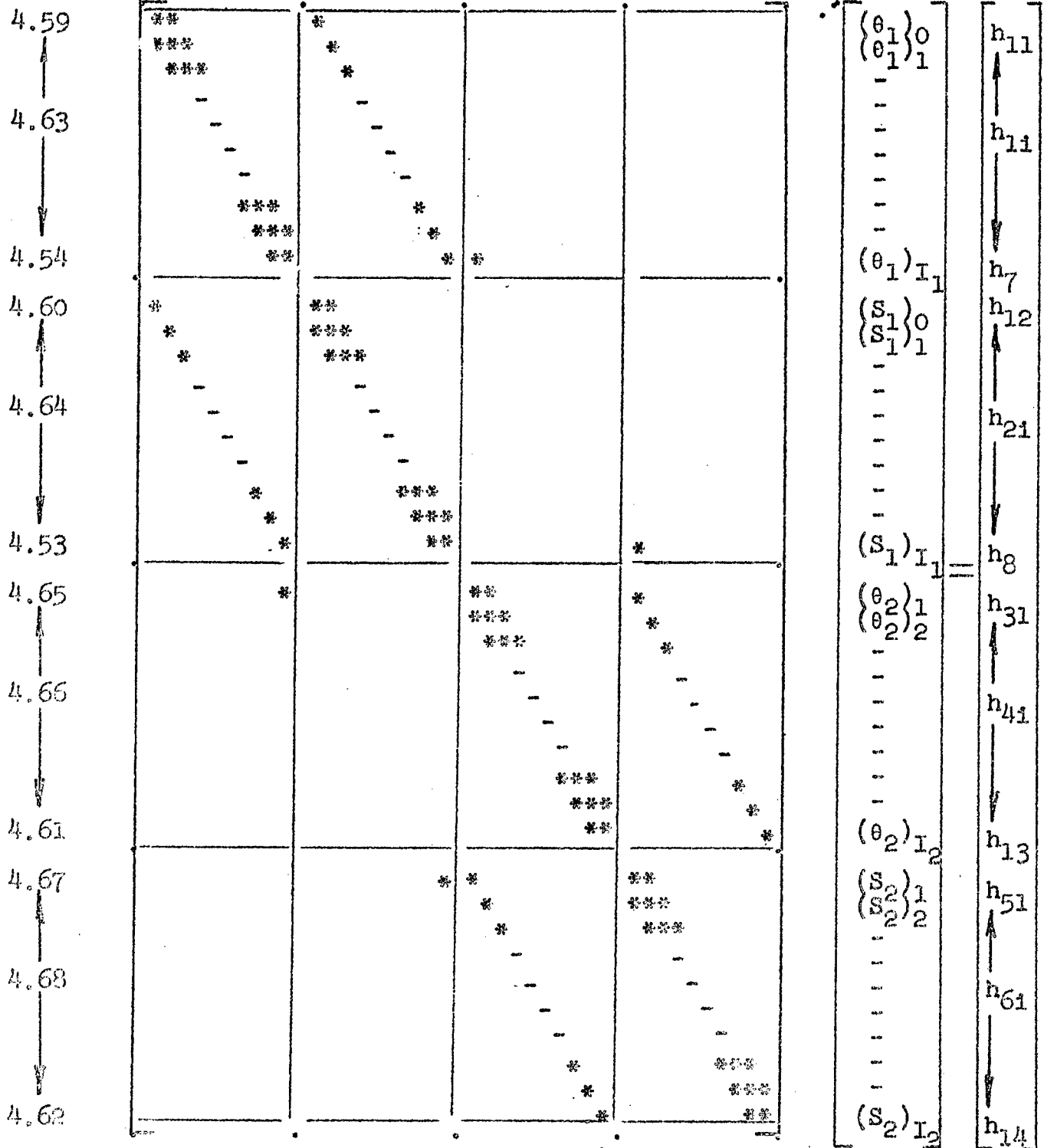
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4.39  
↑  
4.41  
↓  
4.31  
4.40  
↑  
4.42  
↓  
4.32  
4.37  
4.45  
↓  
4.25  
4.38  
4.44  
↓  
4.26



$\theta_1$   
 $\theta_2$   
---  
 $\theta_1$   
 $s_1$   
 $s_2$   
---  
 $s_1$   
 $\phi_1$   
 $\phi_2$   
---  
 $\phi_j$   
 $\psi_1$   
 $\psi_2$   
---  
 $\psi_j$

$k_{17}$   
↓  
 $k_{11}$   
↓  
 $k_{19}$   
 $k_{11}$   
 $k_{18}$   
↓  
 $k_{21}$   
↓  
 $k_{20}$   
 $k_{12}$   
 $k_{15}$   
↓  
 $k_{3j}$   
↓  
 $k_7$   
↓  
 $k_{16}$   
↓  
 $k_{4j}$   
↓  
 $k_8$



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